10 Years BADGeometry: Progress and Open Problems in Ehrhart Theory

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Thanks To. . .

    

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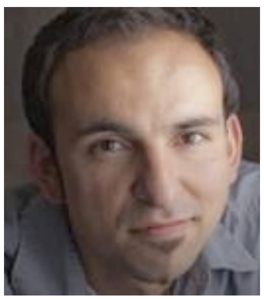






A Warm-Up Example

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Z}^d$

For $t \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right)$,
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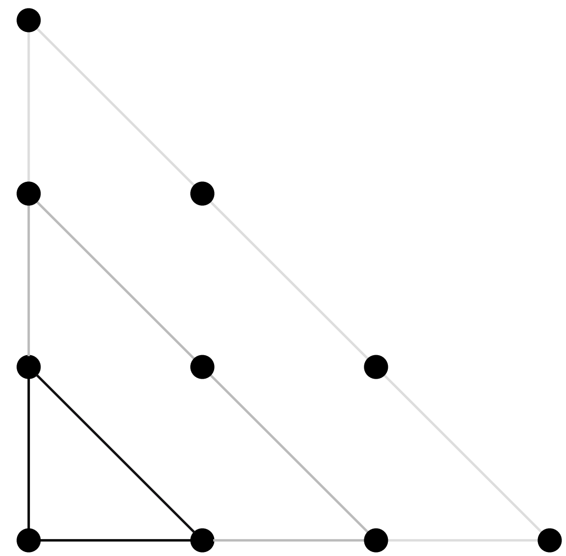

Example:

$$\Delta = \text{conv}\left\{(0,0),(1,0),(0,1)\right\}$$
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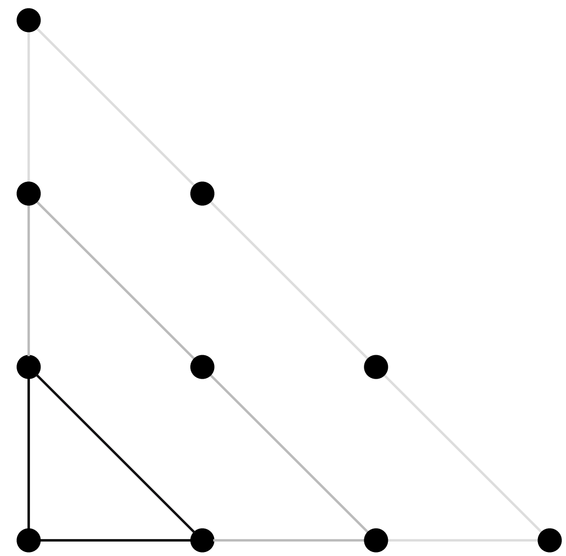

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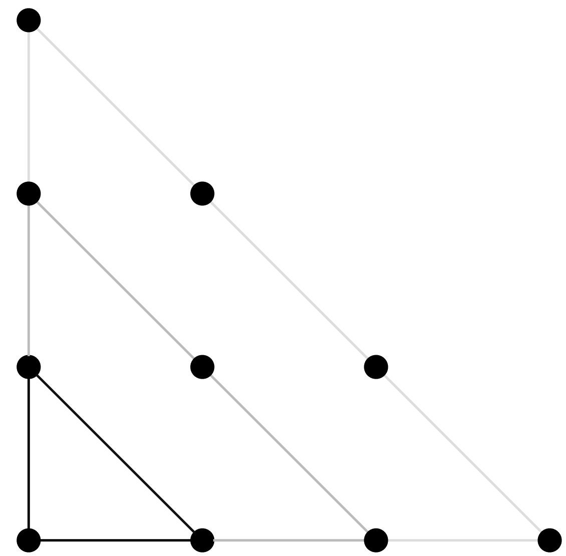

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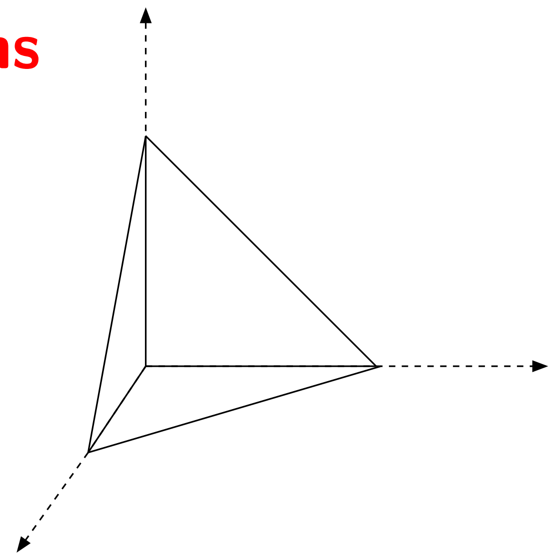
Warm-Up in d Dimensions



The standard d-simplex

$$\begin{aligned}\Delta &= \left\{ \boldsymbol{x} \in \mathbb{R}^d_{\geq 0} : \, x_1 + x_2 + \cdots + x_d \leq 1 \right\} \\ &= \operatorname{conv}\left\{ (0,0,\ldots,0), (1,0,0,\ldots,0), (0,1,0,\ldots,0), \ldots, (0,0,\ldots,0,1) \right\}\end{aligned}$$

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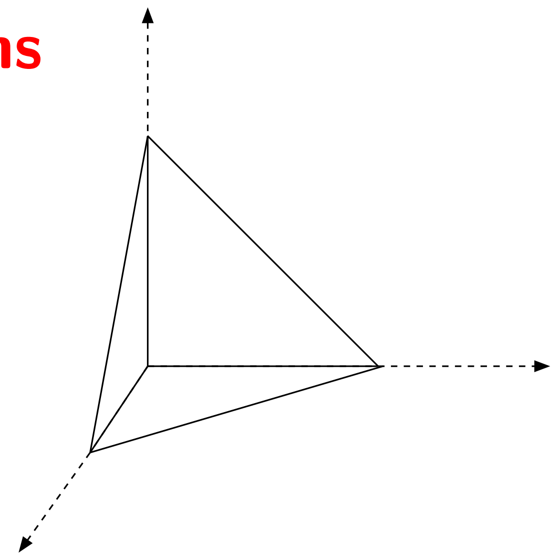

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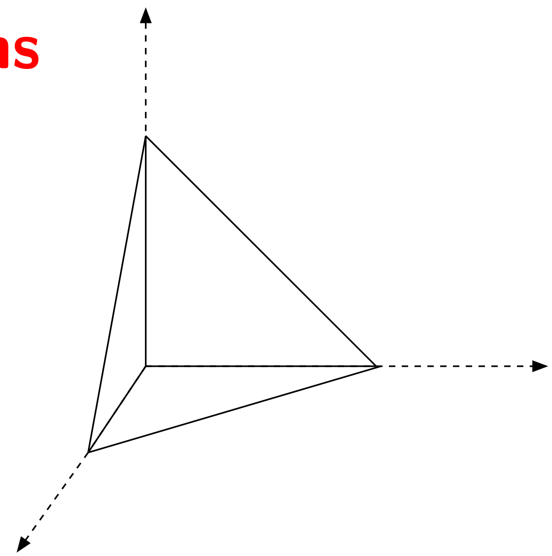

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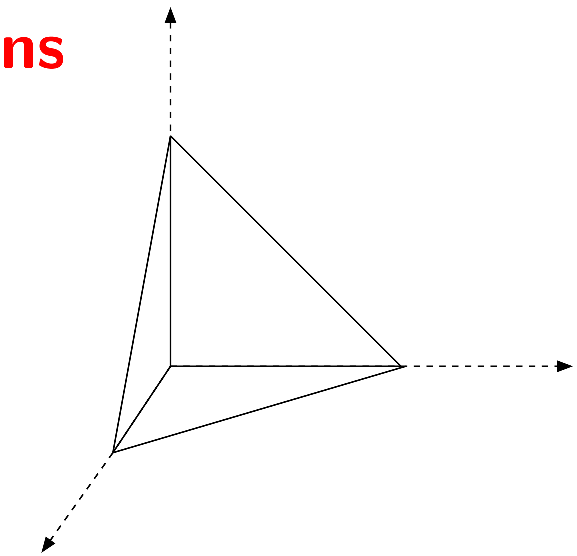

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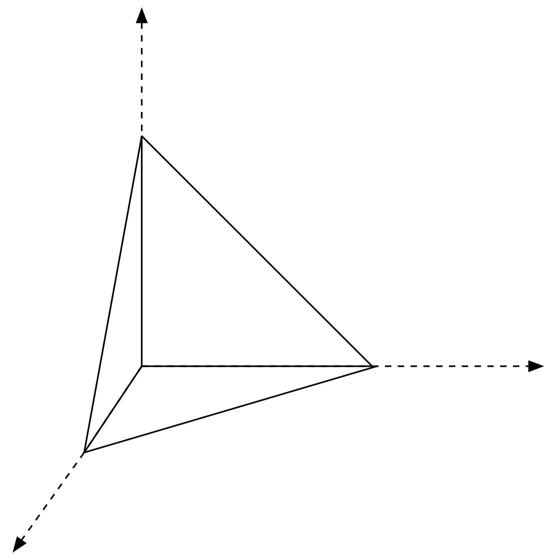

The interior of the standard d-simplex,

$$\Delta^\circ = \left\{ \boldsymbol{x} \in \mathbb{R}^d_{>0} : \, x_1 + x_2 + \cdots + x_d < 1 \right\},$$

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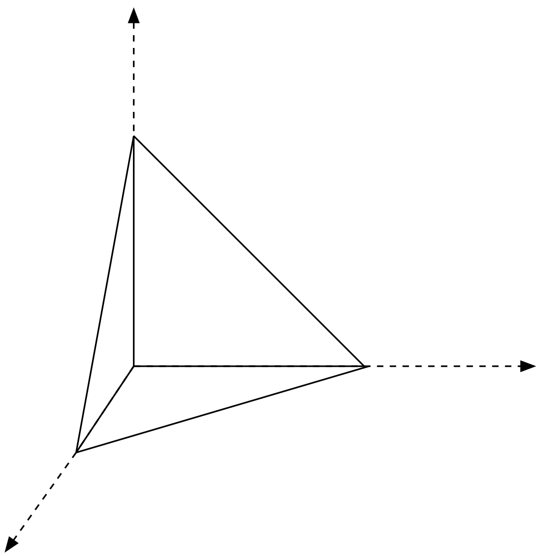

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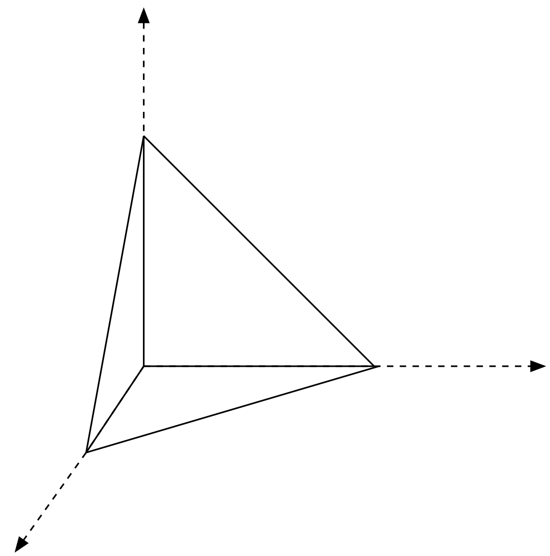

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a polynomial that happens to satisfy the algebraic relation

$$\binom{t-1}{d} = (-1)^d \binom{-t+d}{d}, \qquad \text{that is,} \qquad L_\Delta(-t) = (-1)^d L_{\Delta^\circ}(t).$$

Warm-Up Generating Functions

The discrete volume $L_\Delta(t) = \binom{t+d}{d}$ of the standard d-simplex comes with the friendly generating function

$$\sum_{t \geq 0} \binom{t+d}{d} z^t = \frac{1}{(1-z)^{d+1}} \, .$$

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Motivated by this example, we define the Ehrhart series of the lattice polytope $\mathcal{P}$ as

$$\mathrm{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \ge 1} L_{\mathcal{P}}(t) \, z^t .$$

Ehrhart Polynomials

Theorem (Ehrhart 1962) For any lattice polytope $\mathcal{P}$, $L_{\mathcal{P}}(k)$ is a polynomial in k of degree $\dim \mathcal{P}$ with leading term $\operatorname{vol} \mathcal{P}$ and constant term 1.

Equivalently, $\operatorname{Ehr}_{\mathcal{P}}(z)$ is a rational function of the form

$$\frac{h(z)}{(1-z)^{\dim \mathcal{P}+1}}$$

where the Ehrhart h-vector $h(z)$ satisfies $h(0) = 1$ and $h(1) = (\dim \mathcal{P})! \operatorname{vol} \mathcal{P}$.

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Theorem (Macdonald 1971) $(-1)^{\dim \mathcal{P}} L_{\mathcal{P}}(-k)$ enumerates the interior lattice points in $k\mathcal{P}$.

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- Many discrete problems in various mathematical areas are linear problems, thus they ask for the discrete volume of a polytope in disguise.

- Ehrhart's and Macdonald's theorems allows us to compute a (complicated) integral discreetly (e.g., by interpolating a function at $\frac{d}{2}$ points).

A Few Classic Theorems

Let $\mathcal{P}$ be a lattice d-polytope with Ehrhart h-vector $h(z) = h_s z^s + h_{s-1} z^{s-1} + \cdots + h_0$ (we set all other $h_k = 0$).

Corollary $h_d = \# \left(\mathcal{P}^\circ \cap \mathbb{Z}^d \right)$ and $h_1 = \# \left(\mathcal{P} \cap \mathbb{Z}^d \right) - d - 1$.

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Theorem (Ehrhart) For any rational polytope $\mathcal{P}$, $\mathrm{Ehr}_{\mathcal{P}}(z)$ can be written as $\dfrac{h(z)}{(1 - z^p)^{\dim \mathcal{P} + 1}}$ where p is the denominator of $\mathcal{P}$.

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Classify Ehrhart polynomials (or, alternatively, Ehrhart h-vectors).

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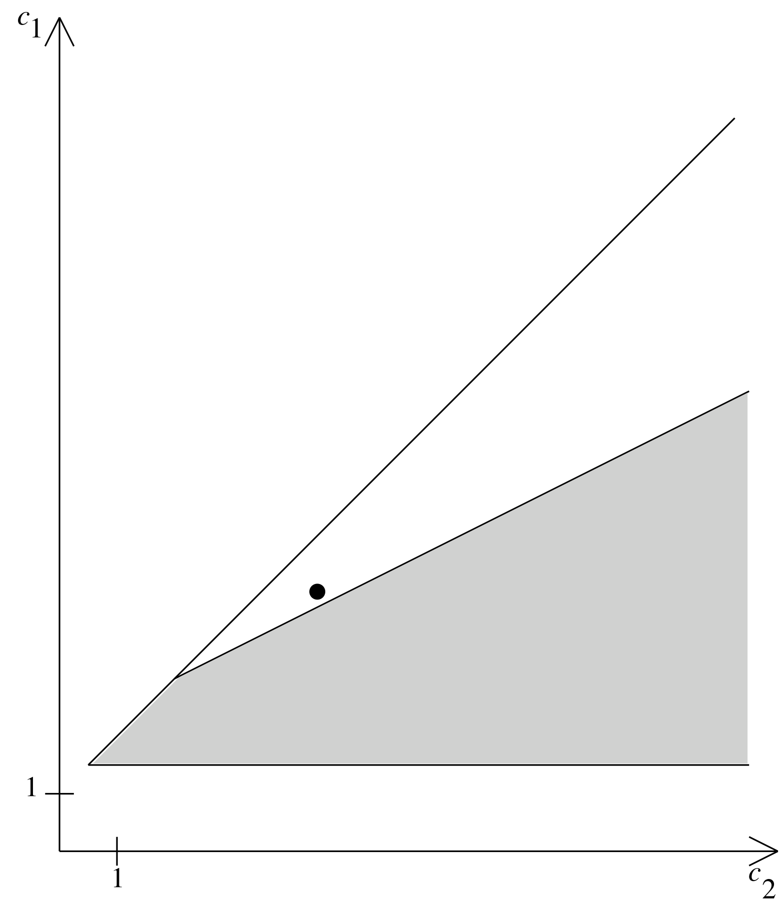
This has been done in dimension ≤ 2 —follows from Pick's Theorem and Scott's inequality for convex lattice polygons (1976).



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Volume Bounds

Let $\mathcal{P}$ be a lattice d-polytope with Ehrhart h-vector

$$h(z) = h_s z^s + h_{s-1} z^{s-1} + \cdots + h_0 \, .$$

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This theorem was conjectured by Batyrev and improves on

Theorem (Lagarias–Ziegler 1991) If $\mathcal{P}$ contains $j \geq 1$ interior lattice points, $\operatorname{vol} \mathcal{P}$ is bounded by a number that depends only on d and j.

Stapledon Decompositions

For a lattice d-polytope with Ehrhart h-vector $h(z)$ of degree s, let $l = d + 1 - s$ be its codegree. (This is the smallest integer such that $l\mathcal{P}$ contains an interior lattice point.)

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Key Observation (Stapledon 2009) There exists a unique decomposition

$$(1 + z + \cdots + z^{l-1})\, h(z) = a(z) + z^l\, b(z)\,,$$

where $a(z) = a_d\, z^d + \cdots + a_0$ and $b(z) = b_{d-l}\, z^{d-l} + \cdots + b_0$ are polynomials with integer coefficients satisfying $a(z) = z^d\, a(\frac{1}{z})$ and $b(z) = z^{d-l}\, b(\frac{1}{z})$.

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Hibi's inequality $h_0 + \cdots + h_{j+1} \geq h_d + \cdots + h_{d-j}$ is equivalent to $a_k \geq 0$,
Stanley's inequality $h_0 + h_1 + \cdots + h_j \leq h_s + h_{s-1} + \cdots + h_{s-j}$ to $b_k \geq 0$.

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The last inequality extends

Theorem (Hibi 1994) If $l = 1$ then $1 \le h_1 \le h_k$ for $2 \le k < d$.

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This machinery yields all possible "balanced" inequalities for Ehrhart h-vectors in dimensions ≤ 6.

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- use a theorem of Payne (2009) on computing Ehrhart h-vectors using a multivariate version of the h-vector of a triangulation
- realize the symmetry in Payne's "boxes"
- use Kneser's Theorem on subsets of abelian groups.

Relaxing I: Unimodular Triangulations

A triangulation τ of $\mathcal{P}$ is unimodular if for any simplex of τ with vertices $v_0, v_1, \dots, v_d$, the vectors $v_1 - v_0, \dots, v_d - v_0$ form a basis of $\mathbb{Z}^d$.

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Reiner–Welker (2005) and Athanasiadis (2005) use this as a starting point to give conditions under which the Ehrhart h-vector is unimodal, i.e.,

$$h_d \le h_{d-1} \le \cdots \le h_j \ge h_{j-1} \ge \cdots \ge h_0 \qquad \text{for some } j.$$

In particular, Athanasiadis proved that the Ehrhart h-vector of the Birkhoff polytope is unimodal (conjectured by Stanley).

Relaxing II: Veronese Constructions

For a lattice d-polytope $\mathcal{P}$ with Ehrhart h-vector $h(z)$, define the Hecke operator U_n through

$$\mathrm{Ehr}_{n\mathcal{P}}(z) = \frac{\mathrm{U}_n \, h(z)}{(1-z)^{d+1}} \, .$$

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Theorem (Brenti–Welker 2008) There exists real numbers $\alpha_1 < \alpha_2 < \cdots < \alpha_{d-1} < \alpha_d = 0$, such that for for any lattice d-polytope $\mathcal{P}$ and n sufficiently large, $\mathrm{U}_n\, h(t)$ has negative real roots $\beta_1(n) < \beta_2(n) < \cdots < \beta_{d-1}(n) < \beta_d(n) < 0$ with $\lim_{n\to\infty} \beta_j(n) = \alpha_j$. Consequently, $\mathrm{U}_n\, h(t)$ is unimodal.

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Theorem (MB–Stapledon 2010) The α_k's are the roots of the Eulerian polynomial, and "sufficiently large" depends only on d. Furthermore, the coefficients $h_0(n), h_1(n), \dots, h_d(n)$ of $\mathrm{U}_n\, h(t)$ satisfy

$$1 = h_0(n) < h_d(n) < h_1(n) < \cdots < h_j(n) < h_{d-j}(n) < h_{j+1}(n) < \cdots < h_{\lfloor \frac{d+1}{2} \rfloor}(n) \, .$$

Special Class I: Cyclic Polytopes & Friends

Fix n lattice points on the moment curve $\nu_d(t) = \left(t, t^2, \dots, t^d\right)$ in $\mathbb{R}^d$ and let $C(n,d)$ be their convex hull (a cyclic polytope).

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Theorem (Liu 2005)

$$L_{C(n,d)}(t) = \operatorname{vol}(C(n,d))\, t^d + L_{C(n,d-1)}(t)\,.$$

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This theorem gave rise to the family of lattice-face polytopes (Liu 2008).

Special Class II: Reflexive Polytopes

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Theorem (Mustata–Payne 2005, Payne 2008). Hibi's conjecture fails in all dimensions ≥ 6. More precisely, for any $m, n > 0$ there exists a reflexive polytope dimension $O(m \log \log n)$ such that $h(z)$ has at least m valleys of depth at least n.

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A lattice polytope $\mathcal{P}$ is normal if every lattice point in $t\mathcal{P}$ is a sum of t lattice points in $\mathcal{P}$.

Open Problem Does Hibi's conjecture hold for normal reflexive polytopes?

Special Class III: Matroid Polytopes

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Matroid polytopes are special cases of generalized permutahedra (Postnikov 2009, Ardila–Postnikov 2010), so there is lots of room to play. . .

If You Ever Want To Compute. . .

YOU should check out Jesús De Loera, Matthias Köppe, et al's LattE

www.math.ucdavis.edu/∼latte

and Sven Verdoolaege's barvinok

freshmeat.net/projects/barvinok

Rational Polytopes

If $\mathcal{P}$ has rational vertices with common denominator p then $L_{\mathcal{P}}(t)$ is a quasipolynomial of degree $\dim \mathcal{P}$ with period p.

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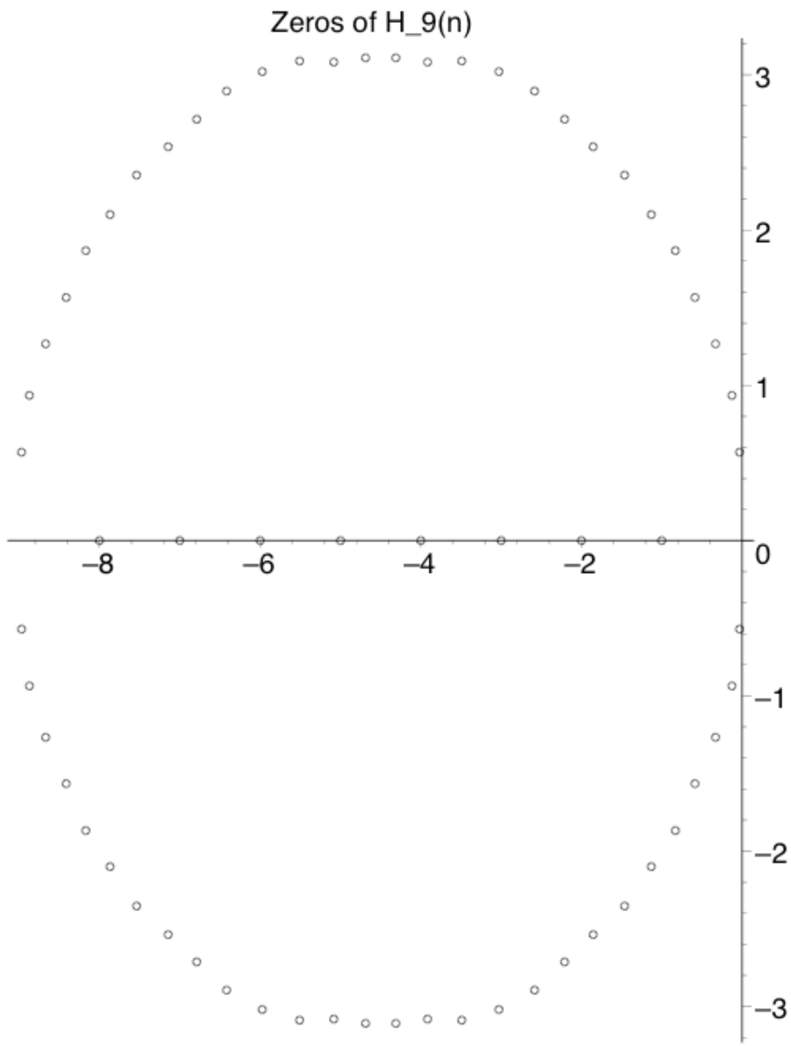
Linke (arXiv:1006.5612) introduced theory of rational dilation where the coefficients of "Ehrhart quasipolynomials" become piecewise polynomial functions.

One Last Picture. . .



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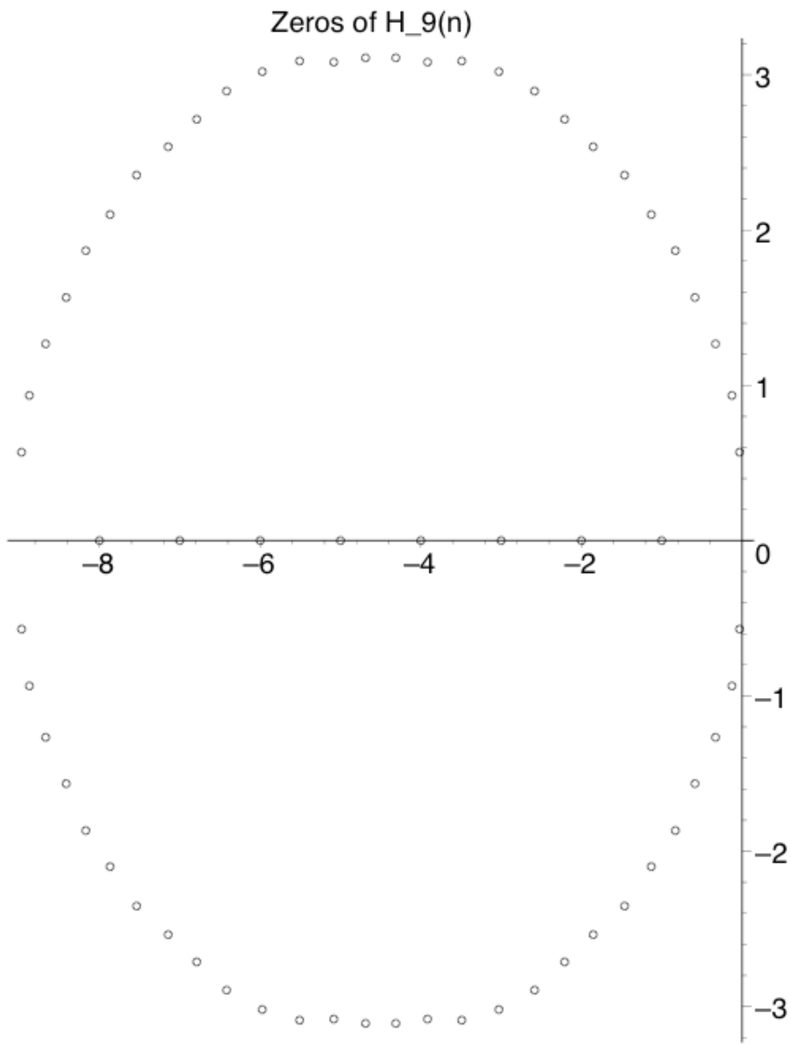

For more about zeros of (Ehrhart) polynomials, see Braun (2008) and Pfeifle (2010).